Fourier Analysis

Mar 28, 2024

Review.

Thm (Fourier Inversion formula)

Let $f \in M(\mathbb{R})$. Suppose that $\hat{f} \in M(\mathbb{R})$.

Then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

Below we first give an application of the Inversion formula.

Thm 1 (Plancherel formula).

Let $f \in M(\mathbb{R})$. Suppose $\hat{f} \in M(\mathbb{R})$.

Then we have

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi.$$

Proof. Let $h(x) = \overline{f(-x)}$. Then $h \in M(\mathbb{R})$.

Notice that

$$\hat{h}(\xi) = \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-\infty}^{\infty} \overline{f(-x) \cdot e^{2\pi i \xi x}}\, dx$$

$$= \overline{\int_{-\infty}^{\infty} f(-x)\, e^{2\pi i \xi x}\, dx}$$

$$\overset{\text{Letting } y=-x}{=} \overline{\int_{\infty}^{-\infty} f(y)\, e^{-2\pi i \xi y} (-)\, dy}$$

$$= \overline{\hat{f}(\xi)}.$$

Let us consider $f * h$.

Notice that $f * h \in \mathcal{M}(\mathbb{R})$.

$$\widehat{f * h}(\xi) = \hat{f}(\xi) \cdot \hat{h}(\xi) = \hat{f}(\xi) \cdot \overline{\hat{f}(\xi)}$$

$$= \left| \hat{f}(\xi) \right|^2$$

Hence $\widehat{f * h} \in \mathcal{M}(\mathbb{R})$.

Now applying Fourier Inversion formula to $f * h$, we obtain

$$f * h(0) = \int_{-\infty}^{\infty} \widehat{f * h}(\xi) \, d\xi$$

$$= \int_{-\infty}^{\infty} \left| \hat{f}(\xi) \right|^2 d\xi .$$

Notice that

$$f * h(0) = \int_{-\infty}^{\infty} f(x) \, h(-x) \, dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot \overline{f(x)}\, dx$$

$$= \int_{-\infty}^{\infty} |f(x)|^2 \, dx .$$

So we obtain

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi .$$

$$\boxed{///}$$

§ 5.5 <u>Application 1</u>: The time-dependent heat equation on the real line.

Consider the heat equation on the line

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0 & ① \\ u(x,0) = f(x), \quad x \in \mathbb{R} \quad (\text{initial condition}) & ② \end{cases},$$

where $u = u(x,t)$ is the temperature at point x and time t.

We first find a solution by a formal argument:

Taking the Fourier transform on both sides of ① (with respect to x).

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} = (2\pi i \xi)^2 \, \hat{u}(\xi,t)$$

$$= -4\pi^2 \xi^2 \, \hat{u}(\xi,t).$$

(Check: $\displaystyle\int_{\mathbb{R}} \frac{\partial u(x,t)}{\partial t} e^{-2\pi i \xi x} dx$

$$= \frac{\partial}{\partial t} \int_{\mathbb{R}} u(x,t) e^{-2\pi i \xi x} dx$$

$$= \frac{\partial}{\partial t} \hat{u}(\xi, t).$$

$$\int_{\mathbb{R}} \frac{\partial^2 u(x,t)}{\partial^2 x} e^{-2\pi i \xi x} dx$$

$$= (2\pi i \xi)^2 \cdot \int_{\mathbb{R}} u(x,t) e^{-2\pi i \xi x} dx$$

$$= -4\pi^2 \xi^2 \hat{u}(\xi, t).$$
)

Hence we obtain a first order Linear ODE

$$\frac{d}{dt} \hat{u}(\xi,t) = -4\pi^2 \xi^2 \hat{u}(\xi,t).$$

Fix ξ, we see that

$$\hat{u}(\xi,t) = A(\xi)\cdot e^{-4\pi^2\xi^2 t}$$

(taking $t=0$)

Observe that from the inital condition ②,

$$\hat{u}(\xi,0) = \hat{f}(\xi)$$

Hence we have $A(\xi) = \hat{f}(\xi)$.

Now we obtain

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cdot e^{-4\pi^2\xi^2 t}$$

Next we define

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}}\, e^{\frac{-x^2}{4t}} \quad \text{for } x\in\mathbb{R} \text{ and } t>0.$$

Check : $\widehat{\mathcal{H}_t}(\xi) = e^{-4\pi^2\xi^2 t}$.

Recall $e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi \xi^2}$.

$$e^{-\frac{x^2}{4t}} = e^{-\pi\left(\frac{x}{\sqrt{4\pi t}}\right)^2} \xrightarrow{\mathcal{F}} \sqrt{4\pi t} \cdot e^{-\pi \xi^2 \cdot (4\pi t)}$$

$$= \sqrt{4\pi t}\, e^{-4\pi^2 \xi^2 t}$$

We call $\{\mathcal{H}_t(x)\}_{t>0}$ the heat kernel on the real line.

By the above analysis, we see that

$$\hat{u}(\xi, t) = \widehat{f * \mathcal{H}_t}(\xi).$$

Now by the inversion formula, we see that

$$\boxed{u(x,t) = f * \mathcal{H}_t(x).}$$